

The Role of Energy and a New Approach to Gravitational Waves in General Relativity

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Abstract

The energy localization hypothesis of the author that energy is localized in non-vanishing regions of the energy-momentum tensor implies that gravitational waves do not carry energy in vacuum. If substantiated, this has significant implications for current research. Support for the hypothesis is provided by a re-examination of Eddington's classic calculation of energy loss by a spinning rod. It is emphasized that Eddington did not monitor the entire Tolman energy integral, concentrating solely upon the change of the 'kinetic' part of the energy. The 'quadrupole formula' is thus seen to measure the kinetic energy change. When the derivative of the missing stress-trace integral is computed, it is seen to cancel the Eddington term and hence the energy of the rod is conserved, in support of the localization hypothesis. The issue of initial and final states is addressed.

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1. Introduction

From the time of its inception more than 80 years ago, general relativity has presented problems in connection with the energy concept. In other areas of physics, there were tensorial conservation laws both locally and globally. However, in general relativity, global energy-momentum conservation was accessible only through the aid of pseudotensors with their attendant ambiguities. Many researchers from the time of Einstein who introduced them, were willing to deal with these less-than-satisfactory constructs, albeit with an underlying sense of unease. While they did present a certain degree of consistency when used in a particular manner (with asymptotic flatness and Cartesian coordinates), the fact that they could be annihilated at will at any given space-time point by the right choice of coordinates, rendered the issue of energy localization in general relativity problematic. Einstein [15] and Eddington [4] accepted the apparently inherent non-localizability of gravitational energy on the basis of the nature of the pseudotensor. Through the years, various authors [11] chose to concentrate their efforts on direct analyses of the motion of bodies in general relativity, motivated at least in part by the inherent ambiguities presented by the energy concept in general relativity.

However, the energy concept is fundamental to physics and the challenge to understand it more clearly within the context of general relativity continued to attract the attention and effort of various researchers [12]. Interestingly, Bondi, one of the prominent pioneers in the field, recently returned to the question, arguing in favour of the necessity for energy localization [22]. Apart from the fundamental importance of the issue, there were practical considerations as well. The elusive gravitational waves, first proposed by Einstein to be emitted by accelerated masses in analogy with electromagnetic waves from accelerated charges in electromagnetism, had escaped all but a conjectured indirect indication of their presence from the period variation of the binary pulsar. If there were to be a direct detection, the fundamental nature of these waves vis-à-vis energy content would have to be understood.

In recent years, various factors led this author to a new hypothesis for the localization of energy in general relativity [1]. The generally covariant energy-momentum conservation laws in general relativity which are devoid of content in vacuum, acquire apparent content in vacuum when re-expressed as an ordinary divergence with the pseudotensor and the Gauss theorem employed. Apparent wave energy is computed by a flux emanating from the pseudotensor in the asymptotic region yet the originating equation is actually devoid of content in vacuum. This suggests that the pseudotensor might be injecting “pseudo-content”, which would not be surprising, given its ephemeral nature. Plane gravitational waves are traditionally expressed in the “transverse- traceless” gauge in which the pseudotensor reveals an apparent energy flux. However, since such waves are in the Kerr-Schild class, they can be expressed in a form in which all components of their pseudotensor vanish globally [10]. Then they indicate a total lack of energy or momentum, unlike their electromagnetic plane-wave cousins which have a tensorially covariant energy and momentum content. Particle physicists generally tend to view gravity as just another field and the graviton as just another zero-rest-mass particle. However, it is to be emphasized that all particles and fields apart from gravity exist *within* spacetime whereas, in essence, gravity *is* spacetime. From the general relativity perspective, gravity assumes a very special role. These and other facts [1] led the author to hypothesize that *in generality, energy and momentum are localized in regions of the energy-momentum tensor T_i^k* . This would imply that *gravitational waves are not carriers of energy and momentum in vacuum* [1]. If correct, this would have far-reaching consequences. Apart from a revamping of basic concepts in classical general relativity, the nature of the quantization of gravity would have to be re-evaluated: for what is a graviton in the absence of energy? Various earlier demonstrations to the contrary of the hypothesis were analyzed and their flaws indicated [2] but it was evident that a clear-cut demonstration in favour of the hypothesis was called for. This is the primary aim of the present paper.

To this end, we focus upon one of the very earliest calculations of supposed energy loss by a dynamical system in the form of a spinning rod [4]. We first show that the complete mass function was not monitored by this calculation and that a vital missing part [5] is required to do so. We achieve this by a modification of a method devised by Einstein. A remarkable thing emerges when the missing Tolman terms are accounted for: *the supposed energy loss first calculated by Eddington is precisely cancelled by the missing Tolman terms in support of our localization hypothesis.* While we analyzed this particular system because of its direct physical clarity, the result can be generalized in the manner of [8].

In sec. 2, we build the mathematical framework for the theory. In sec. 3, we review the ingenious calculation of Eddington for the spinning rod and in the process, indicate where he had made an unjustified assumption in pre-supposing the negligible character of certain terms. In actuality, these terms which Eddington neglected undergo a fortuitous cancellation. In sec. 4, we review the achievement of Tolman in expressing the energy of a static or quasi-static system as an integral over the region of the energy-momentum tensor. In sec. 5, we use a modification of the method of Einstein to express the contributions to mass which had been neglected previously. These are seen to cancel the original Eddington portion of the mass-loss function. A summary and concluding discussion are presented in sec. 6.

2. Theoretical Framework

The starting point is the field equation of general relativity [13]

$$G_i^k = \frac{8\pi G}{c^4} T_i^k \quad (1)$$

The energy-momentum tensor is the source of the gravitational field, which in turn is embodied in the Einstein tensor G_i^k . While the T_0^0 component encompasses all energy density apart from gravity, the latter certainly affects the energy content of a system. This is most clearly seen from the mass defect of a spherically symmetric ball of matter. We had shown how a neglect of the gravitational contribution can lead to confusion in the tallying of mass [7].

Given the fact that gravity affects the energy of a system, we now consider what role it may play in the localization of the energy. Various authors have presented reasons for regarding the energy, inclusive of the contribution from gravity, in a spherically-symmetric system to be within the region of T_i^k [12] although some have argued that it is actually distributed in the entire field [14]. If it were to be the latter, then the

gravitational field would be seen to have an energy distribution very much like that of the electromagnetic field. However, we will see below that there are reasons to believe otherwise.

It is the conservation laws which lend fundamental significance to the energy concept and in covariant form, applicable to general relativity, these are

$$T_{i;k}^k = 0 \quad (2)$$

It is to be noted that this equation has content only in the region of non-vanishing energy-momentum tensor T_i^k : in vacuum, it reduces to the empty identity $0 = 0$. It is when we shift from the local conservation of (2) to a form amenable to global conservation with the introduction of the pseudotensor t_i^k

$$\frac{\partial}{\partial x^k} \left(\sqrt{-g} (T_i^k + t_i^k) \right) = 0 \quad (3)$$

in the form of an ordinary vanishing divergence that the unique features of general relativistic energy considerations become manifest.

Unlike T_i^k , the pseudotensor, which is constructed from the first partial derivatives of the metric tensor, can be annihilated at any pre-assigned point and it is this feature which has led probably most researchers from Einstein to the present to deny the possibility of a logical energy localization for general relativity. However, some authors have come to the position that energy can be localized in the T_i^k regions for spherically symmetric sources and some have extended this to more general static or stationary sources [12].

Many papers have been written which emphasize the similarities between electromagnetism and linearized general relativity. However, there are important differences which tend to get overlooked. For example, plane gravitational waves are seen to be transverse in essence as are plane electromagnetic waves and the energy- and momentum-carrying property as seen by the pseudotensor in the gravitational case is held in analogy with the energy momentum tensor in the electromagnetic case. However, whereas the latter has a generally covariant character, the former can not only be annihilated at any pre-determined point, as is well-known but also, as is less well-known, it can be expressed in Kerr- Schild form in which all components of the pseudotensor vanish globally. Thus, while the energy and momentum of electromagnetic waves is indisputable both theoretically and from solid experimental evidence, the situation for gravity waves is clearly not on the same footing.

Another example is that of a spherical ball of charge as compared to a spherical ball of mass. In the former case, the energy density as read from T_{00} component of the covariant energy-momentum tensor, is distributed throughout space whereas in the

latter case, there is support for the localization of the energy of the system within the region of T_{ik} , in this case, within the ball itself.

These considerations led the author to the localization hypothesis [1]. An immediate consequence is that gravitational waves would not be carriers of energy in vacuum. This is in contradiction to many previous calculations which have attributed an energy loss to systems which emit gravitational waves and the generally prevailing belief.

Einstein [15](see also [6]) used the pseudotensor in the context of linearized general relativity for weak gravitational fields

$$g_{ik} = \eta_{ik} + h_{ik} \quad (4)$$

(η_{ik} is the Minkowski metric and h_{ik} is a small perturbation) to demonstrate that a system with a time-varying mass-quadrupole tensor $D_{\alpha\beta}$ would lose energy at the rate given by the “quadrupole formula”

$$\dot{E} = -\frac{G}{45c^5} \left(\frac{d^3}{dt^3} \right) D_{\alpha\beta} \left(\frac{d^3}{dt^3} \right) D^{\alpha\beta} \quad (5)$$

This was almost universally accepted and employed through the years, particularly in conjunction with systems driven dynamically by non-gravitational forces. Challenges were essentially confined to systems such as binary stars which are driven by gravity itself [3].

After Einstein’s calculation, Eddington confirmed the result for the specific system of a spinning rod [4] by a direct calculation with the fields and the pseudotensor. Later, he performed the analogue of an electromagnetic radiation damping calculation for the spinning rod and confirmed both his previous result and the quadrupole formula (5). This will be the focus of the following section.

3. Eddington’s Spinning Rod

Einstein used the Gauss theorem in conjunction with (3) to express the the expected rate of change of the energy in a region in terms of the flux of the pseudotensor over the infinite sphere

$$\frac{\partial}{\partial t} \int (\sqrt{-g}(T_0^0 + t_0^0)) dV = -c \oint (\sqrt{-g}t_0^\alpha) dS_\alpha \quad (6)$$

This is similar to the familiar procedure in electromagnetism where the equation for the radiated energy is

$$\frac{\partial}{\partial t} \int ((T_0^0) dV = -c \oint (T_0^\alpha) dS_\alpha \quad (7)$$

However, there are important differences: in (7), there is no pseudotensor and the flux arises from the *tensorial* Poynting vector.

Einstein used (6) to deduce the quadrupole formula (5) for the presumed energy loss by a system with a time-varying mass-quadrupole tensor in analogy with electric quadrupole radiation arising via (7) in Maxwell theory. Eddington chose to focus upon a specific source, a uniform rod of mass m , length $2a$, spinning with angular frequency ω in the x - y plane. He solved for the gravitational field perturbation h_{ik} and used (6) to compute an energy flux

$$\frac{dE}{dt} = -\frac{32GI^2\omega^6}{5c^5} \quad (8)$$

This result agrees with that computed directly from (5). Later, in the second edition of his book [4], Eddington returned to the spinning rod and performed the analogue of an electromagnetic radiation damping calculation using an alternative form of the covariant conservation laws (2) (\mathcal{T}^{ab} is defined as $\sqrt{-g}T$)

$$\frac{\partial}{\partial x^k} (\mathcal{T}_i^k) = \frac{1}{2} \mathcal{T}^{ab} \frac{\partial h_{ab}}{\partial x^i} \quad (9)$$

where (4) was used. For the energy calculation, i is set to 0 and the equation is integrated over the region just beyond the confines of the the source

$$\frac{\partial}{\partial t} \int \mathcal{T}_0^0 dV = \frac{1}{2} \int \mathcal{T}^{ab} \frac{\partial}{\partial t} h_{ab} dV \quad (10)$$

(note that the 3-divergence contribution from the left hand side of (9) gives no contribution upon integration because of Gauss' theorem and the choice of integration volume).

Unlike the previous calculations, this approach has the advantage of avoiding the pseudotensor entirely and there are no asymptotic conditions to confront. Moreover, it mirrors the familiar radiation damping calculations of electromagnetism which had proved so successful in unifying the understanding of the role of energy in electromagnetic wave analysis. Using the linearized form of the Einstein equations (1) for weak gravitational fields in the harmonic gauge, Eddington expressed the retarded integral solution

$$h_{ab} = -4 \int \left[\frac{T'_{ab} - \frac{1}{2} \eta_{ab} T'}{\left(r \left(1 - \frac{v_r}{c} \right) \right)_{ret}} \right] dV' \quad (11)$$

(henceforth, square brackets will indicate retardation by $t - \frac{r}{c}$) in the manner of Lienard-Wiechert, which he substituted into (10). He noted that the resulting integral, now in terms of T^{ab} and T'_{ab} , “exhibits the loss of energy as arising from the mutual action

of pairs of elements of the rod, dV and dV'' . To bring all elements of the resulting expression to a common time t , he performed a present-time expansion

$$\left[\frac{T'_{ab}}{r \left(1 - \frac{v_r}{c} \right)} \right]_{ret} = \frac{T'_{ab}}{r} - \frac{d}{dt} T'_{ab} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dt^n} (r^{n-1} T'_{ab}). \quad (12)$$

of the retarded part. Using equations (12) and (11) in (10), Eddington had a workable form with which to compute the time rate of change of the \mathcal{T}_0^0 integral. For Eddington, this was the energy loss. We will have more to say about this later. For convenience, he chose to evaluate it at $t=0$ with the rod, which is spinning in the x-y, plane being along the x axis at $t=0$. With dV at x and dV' at x' , the distance between elements is

$$r = \sqrt{x^2 + x'^2 - 2xx' \cos \omega t} \quad (13)$$

This is the r to be used in (12) and t set to 0 after the differentiations. Simplifications occur because if T^{ab} is any non-vanishing component at $t = 0$, T'_{ab} will be an even function of time so odd order derivatives vanish. Therefore, he only had to deal with the series

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{T'_{ab}}{r(1 - \frac{v_r}{c})} \right] &= -\frac{d^2}{dt^2} T'_{ab} - \frac{1}{6} \frac{d^4}{dt^4} (T'_{ab} (x^2 + x'^2 - 2xx' \cos \omega t)) \\ &\quad - \frac{1}{120} \frac{d^6}{dt^6} (T'_{ab} (x^2 + x'^2 - 2xx' \cos \omega t)^2) + \dots \end{aligned} \quad (14)$$

which is used with (11) in (10) and the leading terms retained. Further simplifications are realized because the rod is symmetrical about the origin and hence in the integration, all terms which are of odd power in either x or x' will not contribute. He found the contributions to (10) from, respectively, the stress components T^{11} , T^{22} , the momentum components T^{20} , T^{02} and the “energy components” T^{00} , T , the last being the trace T_k^k of the energy-momentum tensor, which Eddington referred to as the “proper-density”. He calculated with T as if it were T^{00} with the rationale that they “are practically the same”. However, this is unjustified because the two quantities differ by the trace of the stress terms ($T_1^1 + T_2^2$) (note that T_3^3 is zero for spin in the x-y plane) and Eddington had already demonstrated that stress component products yield contributions to (10) of the required lowest order where they appear directly, i.e., outside of the trace T terms. Thus, the contributions from the stress terms in the trace, T , must be confronted. However, a simple calculation reveals that both the extra terms which arise as products of stress components with each other as well as the extra terms which arise as products of stress

components with T^{00} cancel each other. Thus Eddington was fortunate in his neglect of these terms [17].

Upon gathering the various contributions, Eddington found in now the third manner of calculation, the expected result (8). Apart from the issue of whether or not such calculations would be adequate in the case in which the rotation is supported by gravity itself, as in a binary system of masses, as opposed to the much stronger cohesive forces in a solid continuous distribution of mass such as in the rod, most researchers were content with the description of radiated gravitational energy loss described above. Indeed it was Eddington [4] who first recognized that for gravitationally bound systems, $\frac{Gm}{c^2 a}$ is of the order $\frac{v^2}{c^2}$ and the linearized approximation is no longer adequate. However, it was the type of calculation by Eddington described above which provided the bedrock of support and faith in the common wisdom that an accelerating system of masses will, in generality, emit energy via gravitational waves in direct analogy to the situation which prevails in electromagnetism. It was the energy localization hypothesis which induced the present author to scrutinize the standard view, for if energy could flow through vacuum regions as in the case of a spinning rod shedding energy as in (8), the localization hypothesis would be untenable.

It is the Tolman integral which changes the standard picture.

4. The Tolman Integral

Using the pseudotensor, Tolman [5] showed that for a static or quasi- static system, the mass could be expressed as

$$E = \int (\mathcal{T}_0^0 - \mathcal{T}_\alpha^\alpha) dV \quad (15)$$

It is to be noted that the work of Tolman *followed* that of Eddington. Landau and Lifshitz [6] derived (15) without recourse to the pseudotensor. They proceeded from the identity, valid for time-independent systems

$$R_0^0 = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{i0} \Gamma_{0i}^\alpha) \quad (16)$$

where Γ_{bc}^a denotes the Christoffel symbol. They integrated (16) over 3-space and applied Gauss' theorem to convert to a surface integral which was evaluated asymptotically with the approximate metric functions

$$g_{00} = 1 - \frac{2Gm}{c^2 r}, g_{\alpha\beta} = -\delta_{\alpha\beta} - \frac{2Gm n_\alpha n_\beta}{c^2 r}, g_{0\alpha} = 0 \quad (17)$$

where n_α is the unit normal. Evaluating the surface integral yields

$$\int R_0^0 \sqrt{-g} dV = \frac{4\pi Gm}{c^2} \quad (18)$$

Finally, from the field equations (1), the Ricci tensor component R_0^0 can be expressed as

$$R_0^0 = \frac{8\pi G}{c^4} (T_0^0 - \frac{1}{2}T) = \frac{4\pi G}{c^4} (T_0^0 - T_\alpha^\alpha) \quad (19)$$

and (15) follows. There are two points to note: firstly, Tolman succeeded in expressing the total energy in the stationary or quasi-stationary state as an integral which is confined to the region of T_{ik} . This has a particular attraction for us in light of the localization hypothesis. Moreover, there are various grounds to support such a localization for the stationary case [12]. Secondly, it is seen that the stresses play a role in the total mass function. In weak sources, they would appear to play a negligible role relative to T_0^0 and in any event, they appear here in conjunction with a mass measure apparently applicable to at most quasi-stationary sources. Hence one might conclude that they are doubly irrelevant for weak-field radiative energy calculations of the form studied by Eddington. We would conjecture that if ever the thought of a role for the stress terms might have been entertained in the past in this connection, it would have been summarily dismissed for these reasons. However, *it is not the magnitude of the stress terms relative to the T_0^0 term which is relevant here but rather its time rate of change*. Also, while it is true that the Tolman integral clearly measures the mass in stationary or quasi-stationary systems, we will see how it can be used to extract information in dynamic systems as well.

5. Mass Loss in Dynamic Systems

While the Tolman integral measures mass in non-dynamic systems, at issue was the problem of finding a mass measure for dynamic systems. Bondi [20] derived a “news function” to this end but Madore and we [19] showed that this was actually equivalent to the pseudotensor and hence shares in its maladies. It is to be noted that Bondi was careful to state that his mass, which is the mean value of his “mass aspect” function is *defined* as the mass [20]. That it is a definition is frequently overlooked. In any event, Bondi as well as Bonnor [21] and Feynman (in a private communication) shared the view that the ideal situation would be the following: monitor a system from an initial stationary state through a dynamical phase and ending in a final stationary state. The mass

difference between the initial and final states would reveal the unambiguous mass loss. This approach was advocated precisely because one was confronting an essentially different non-tensorial structure in general relativity and doubts had certainly been raised through the years regarding the very foundations of the standard energy scenario. As well, there certainly was no existing experimentation available to support the theoretical deductions.

A more direct approach is available from the vantage point of one who is leaning towards the localization hypothesis. Since the Tolman integral measures the mass without ambiguity in the initial and final stationary states, the Tolman integral must change in the intermediate dynamic phase if there is to be a mass loss. This is a *necessary* condition [9]. Thus, the test is one of measuring the change in the Tolman integral in the dynamic phase to determine whether the necessary condition for mass loss is met. We do so for the Eddington rod.

Eddington had already computed the \mathcal{T}_0^0 part of the Tolman integral. To evaluate the trace of the stresses portion, we will use a variation of the method of Einstein [15]. A more elaborate sequence of steps than that of Einstein is now required because of the demand for greater accuracy in the present context. To this end, we first multiply (9) by g^{il} , re-express the derivative as a product and apply the raising operation to get [18]

$$\mathcal{T}_{,k}^{lk} = F^l \quad (20)$$

where

$$F^l = \frac{1}{2} \mathcal{T}^{ab} h_{ab,i} g^{il} + g_{,k}^{il} \mathcal{T}_i^k \quad (21)$$

We now set $l = \gamma$ and multiply by x^δ

$$x^\delta \mathcal{T}_0^{\gamma 0} = -(\mathcal{T}^{\gamma\beta} x^\delta)_{,\beta} + \mathcal{T}^{\gamma\delta} + F^\gamma x^\delta \quad (22)$$

Interchanging γ and δ , adding the equations, integrating over the source region and applying Gauss' theorem to the 3-divergence terms yields

$$\begin{aligned} \frac{\partial}{\partial x^0} \int (\mathcal{T}^{\delta 0} x^\gamma + \mathcal{T}^{\gamma 0} x^\delta) dV &= 2 \int \mathcal{T}^{\gamma\delta} dV + \int (\mathcal{T}^{\gamma 0} v^\delta + \mathcal{T}^{\delta 0} v^\gamma) dV \\ &+ \int (F^\delta x^\gamma + F^\gamma x^\delta) dV \end{aligned} \quad (23)$$

where v^δ is $\frac{dx^\delta}{dt}$

Setting $l = 0$ in (20), multiplying by $x^\gamma x^\delta$, integrating and again applying Gauss' theorem to the 3-divergence term yields

$$\begin{aligned} \frac{\partial}{\partial x^0} \int \mathcal{T}^{00} x^\gamma x^\delta dV &= \int (\mathcal{T}^{0\delta} x^\gamma + \mathcal{T}^{0\gamma} x^\delta) dV \\ &+ \int \mathcal{T} (x^\gamma v^\delta + x^\delta v^\gamma) dV + \int F^0 x^\gamma x^\delta dV \end{aligned} \quad (24)$$

After taking $\frac{\partial}{\partial x^0}$ of (24), eliminating $\frac{\partial}{\partial x^0} \int (\mathcal{T}^{\gamma 0} x^\delta + \mathcal{T}^{\delta 0} x^\gamma) dV$ by using (23) and setting $\delta = \gamma$ yields

$$\begin{aligned} \int \mathcal{T}^{\gamma\gamma} dV &= \frac{1}{2} \frac{\partial^2}{\partial (x^0)^2} \int \mathcal{T}^{00} x^\gamma x^\gamma dV - \int F^\gamma x^\gamma dV - \frac{1}{2} \frac{\partial}{\partial x^0} \int F^0 x^\gamma x^\gamma dV \\ &- \frac{\partial}{\partial x^0} \int \mathcal{T}^{00} x^\gamma v^\gamma dV - \int \mathcal{T}^{\gamma 0} v^\gamma dV \end{aligned} \quad (25)$$

To complete the energy loss calculation, we require the time rate of change of the integrated trace of the stresses. This is readily connected to (25) as follows:

$$\begin{aligned} \frac{\partial}{\partial x^0} \int \mathcal{T}_\gamma^\gamma dV &= \frac{\partial}{\partial x^0} \int \mathcal{T}^{\gamma k} g_{k\gamma} dV = \int T_{,0}^{\gamma\beta} g_{\beta\gamma} dV + \int \mathcal{T}_{,0}^{\gamma 0} g_{0\gamma} dV + \int \mathcal{T}^{\gamma k} g_{k\gamma,0} dV \\ &= -\frac{\partial}{\partial x^0} \int \mathcal{T}^{\gamma\gamma} dV + \int T_{,0}^{\gamma\beta} h_{\beta\gamma} dV + \int \mathcal{T}_{,0}^{\gamma 0} h_{0\gamma} dV \\ &+ \int \mathcal{T}^{\gamma 0} h_{0\gamma,0} dV + \int \mathcal{T}^{\gamma\beta} h_{\beta\gamma,0} dV \end{aligned} \quad (26)$$

where (4) and $\eta_{\alpha\beta} = \text{diagonal}(-1, -1, -1)$ have been used.

Finally, with (25) substituted into (26) and simplified, we get the final required expression

$$\begin{aligned} \frac{\partial}{\partial x^0} \int \mathcal{T}_\gamma^\gamma dV &= -\frac{1}{2} \frac{\partial^3}{\partial (x^0)^3} \int \mathcal{T}^{00} x^\alpha x^\alpha dV + \frac{\partial}{\partial x^0} \int F^\alpha x^\alpha dV \\ &+ \frac{1}{2} \frac{\partial^2}{\partial (x^0)^2} \int F^0 x^\alpha x^\alpha dV + \frac{\partial^2}{\partial (x^0)^2} \int \mathcal{T}^{00} x^\alpha v^\alpha dV \\ &+ \frac{\partial}{\partial x^0} \int \mathcal{T}^{\alpha 0} v^\alpha dV + \int \mathcal{T}_{,0}^{\alpha k} h_{\alpha k} dV + \int \mathcal{T}^{\alpha k} h_{k\alpha,0} dV \end{aligned} \quad (27)$$

where F^l is given by (21) and

$$g^{il} = \eta^{il} - h^{il} \quad (28)$$

will be used in conjunction with (27) and (21).

The complete energy-momentum tensor to lowest order for the spinning rod is required. At $t = 0$ when the rod is aligned along the x axis, Eddington [4] has supplied all but the T_{11} stress component. This is derived from elementary dynamics with the boundary condition that the tension vanish at the extremities,

$$T_{11} = \frac{\sigma\omega^2}{2}(x^2 - a^2) \quad (29)$$

where σ is the line mass density $\frac{m}{2a}$. From a rotation given by the orthogonal transformation,

$$\begin{aligned} x' &= x\cos\omega t - y\sin\omega t \\ y' &= x\sin\omega t + y\cos\omega t \end{aligned} \quad (30)$$

the complete time-dependent T'^{ik} follows:

$$\begin{aligned} T'^{11} &= \frac{\sigma\omega^2}{2}(x^2 - a^2)\cos^2\omega t + \sigma\omega^2 x^2 \sin^2\omega t \\ T'^{22} &= \frac{\sigma\omega^2}{2}(x^2 - a^2)\sin^2\omega t + \sigma\omega^2 x^2 \cos^2\omega t \\ T'^{12} &= -\frac{\sigma\omega^2}{4}(x^2 + a^2)\sin 2\omega t \\ T'^{01} &= -\sigma\omega x \sin\omega t \\ T'^{02} &= \sigma\omega x \cos\omega t \\ T'^{00} &= \sigma \end{aligned} \quad (31)$$

It is to be noted that the x axis is the coincident axis with the rod at $t = 0$ and the x' axis is the coincident axis with the rod at the instant of time t . Neither axis is co-moving with the rod. Using (31) and (30), it is readily demonstrated that the lowest order conservation laws

$$T'_{,k}^{ik} = 0 \quad (32)$$

are satisfied, as required. The calculation of the Tolman term of (27) now proceeds using the terms of the energy-momentum tensor of (31). It is clear that this calculation is considerably more complex than that of Eddington as in the latter, there was only the

single integral (which leads to the various factors with summation on a and b) given in (10). By contrast, we have many integrals to compute for the Tolman contribution as seen in (27). Moreover, for Eddington, the energy-momentum tensor occurs without a derivative whereas we have to compute terms with derivatives of the energy-momentum tensor. It was for this reason that we required more detailed information regarding the energy-momentum tensor than was required by Eddington. However, the basic procedure is the same as that in the Eddington calculation.

The Gauss theorem is used frequently in the course of the calculations and time derivatives of integrals are evaluated with the differentiation brought into the integral. This is straightforward if there is continuity but in the present context, we are dealing with the discontinuous distribution of a line mass. We had dealt with the changes which occur for sources with discontinuities [23] and presented a more detailed version of the quadrupole formula for such sources. Regarding discontinuities in the present problem, we considered the correction to the Gauss theorem

$$\int dV \nabla \cdot \mathbf{F} = \oint_S d\mathbf{S} \cdot \mathbf{F} + \oint_D d\mathbf{S} \cdot \mathbf{F}|, \quad (33)$$

the operation of differentiating within an integral of a function f ,

$$\frac{d}{dt} \int dV f = \int dV f,0 + \oint_D d\mathbf{S} \cdot \mathbf{v} f|, \quad (34)$$

and when further derivatives are called for,

$$\frac{d}{dt} \oint_D d\mathbf{S} \cdot \mathbf{F}| = \oint_D d\mathbf{S} \cdot \frac{d\mathbf{F}}{dt}| + \oint_D d\mathbf{S} \cdot (\mathbf{F}(\nabla \cdot \mathbf{v}) - (\mathbf{F} \cdot \nabla) \mathbf{v}) \quad (35)$$

where “ $|$ ” denotes inner-minus-outer and \mathbf{v} is the velocity of the surface of discontinuity.

Very rarely are these refinements considered in the literature. However surely there is a virtue in completeness, particularly in the present case where such a major departure from the prevailing beliefs is being presented. We analyzed the various points in the calculation with the extra contributions arising from the presence of discontinuities and found that they contribute at most higher order corrections.

Thus, we are left with the calculation of the derivative of the Tolman stress trace integral (27) using the energy-momentum tensor components (31), the metric and its derivatives using (11), (12), (13), (14), (28) and the symmetries discussed previously. Clearly, there are numerous terms to consider, far more than in the case of the Eddington calculation. We had to deal with the gradient of the stresses, which do not enter into the Eddington calculation. Moreover, the trace T' in (11) as well as the determinant of the

metric were taken into account wherever they appeared. While most of the calculation is routine, there are interesting aspects. For example, in the course of the derivation of the ‘quadrupole formula’ ([15] [6]), the derivatives of the spatial components of the energy-momentum components are related to the derivatives of the mass quadrupole moments using the untraced form of (27) with only the first term on the right hand side retained. However, for our calculation, we require the 3-trace and since $x^\alpha x^\alpha$ is simply the constant r^2 of each element of the rotating rod, we get no contribution from this corresponding term. In passing, we remark that if ever one might be inclined to dismiss the role of stress terms in the tallying of mass loss in general relativistic calculations, one need only remind one’s self as to the origin of the quadrupole formula itself, with the stresses entering as just outlined. Moreover, we are reminded of the importance of the stresses in computing the total energy of a body [7] [16].

For the present calculation, it is from the other terms of (27) that the non-vanishing contribution is found. After the they are summed, a remarkable result emerges: *The derivative of the integrated trace of the stresses is precisely the value (8) that was found by Eddington [4] for the rate of change of the ‘kinetic energy’ \mathcal{T}_0^0 integral and hence the complete Tolman integral does not change in the dynamic phase to lowest (quadrupole formula) order. The mass is conserved to at least this order.*

7. Summary and Concluding Discussion

We began by outlining the unusual role that energy has played in general relativity and the various ideas which have been expressed through the years regarding the issue of its localizability. The reasons which led the present author to hypothesize that energy is most logically localized in regions of non-vanishing energy-momentum tensor T^{ik} were presented. It was noted that if correct, the hypothesis would lead to a hitherto unprecedented aspect of a wave in the case of gravitation: waves carrying real curvature through vacuum would nevertheless be devoid of energy. While such a conclusion might at first glance appear untenable, it was noted that gravity plays a different role in physics from the perspective of general relativity: all particles and fields exist *within* spacetime whereas gravity, in essence, *is* spacetime [1].

We discussed the role of the pseudotensor as a vehicle for computing energy loss by a dynamic system and we noted that plane gravitational waves could be expressed in Kerr-Schild form in which all components of the pseudotensor vanish. This was distinguished from plane electromagnetic waves which have tensorial as opposed to pseudotensorial content and hence display an invariantly significant energy flux. This led us to re-consider

one of the classical calculations in general relativity, the supposed energy loss from a spinning rod as performed by Eddington. It was noted that Eddington found a mass loss consistent with the quadrupole formula via a flux integral with the pseudotensor as well as with a radiation reaction calculation analogous to that which is performed in electromagnetism. However, it was noted that this was done some years prior to the discovery by Tolman that the trace of the stresses, T_α^α , is required in addition to T_0^0 to derive the energy of a stationary or quasi-stationary system in general relativity.

Although a spinning rod is not a stationary system, the results of Tolman could be applied as follows: to counter doubts which had been raised through the years and to trace the evolution of a mass loss without ambiguity, experts had argued repeatedly that ideally one should start with a stationary (or at least quasi-stationary) state for a system which evolves dynamically and returns to an ultimately stationary or quasi-stationary state. The start and finish masses are to be compared and the difference would yield the unambiguous mass loss. Thus we noted that this implies that a necessary condition for a mass loss is that the Tolman integral must vary during the dynamic phase, for otherwise the final tally of the Tolman integral would be the same as the start value. We then noted that Eddington had not computed the rate of change of the Tolman integral. Rather, he had computed the rate of change of the integral of T_0^0 , the ‘kinetic’ energy change. We summarized the elegant manner in which Eddington performed this calculation and we turned our attention to the missing Tolman stress terms. Unfortunately, the latter is far more difficult to compute than the former and hence we devised a simplifying approach based upon Einstein’s classic derivation of the quadrupole formula. In so doing, we found the series of integrals which expresses the rate of change of the Tolman stress-trace integral, which, in conjunction with the Eddington integral, gives the formula for *energy* loss as opposed to *kinetic energy* loss. We found the remarkable result that stress-trace part gives precisely the same result as was found from the Eddington calculation. Since the former appears with a minus sign in the complete Tolman expression, it cancelled the latter. The conclusion was thus that *to the quadrupole formula order, there is no energy loss from a spinning rod, consistent with the localization hypothesis*. With the constancy of the complete Tolman integral, it could then be argued that this integral gives the mass *in generality*, not just in the specialized stationary state. The broader implication of this result was noted, namely that if gravity waves were not energetic, then a re-evaluation of the concept of a graviton, and hence of the quantization of gravity, would be called for.

At this point, it is worth returning to the issue of initial and final states. Since kinetic energy is being lost in the dynamic phase, clearly the process could not have been proceeding from the infinite past. One might posit that the system evolved from a quasi-stationary rotating disk-like configuration with a perturbing element which made

it attain a rod shape and hence enter into a dynamic phase. Then the system loses kinetic energy as described above. The previous analysis depended upon the condition $\frac{v^2}{c^2} \gg \frac{Gm}{c^2a}$ for its validity and hence a more precise calculation would be required to continue the analysis through the eventual slow-rotation phase. One might envisage two ideal final configurations: a) the rod-like structure reverts to its former disk-like shape through a perturbative element, into a quasi-stationary form or b) it remains rod-like. In either case, there is the issue of angular momentum to consider. We would conjecture that just as energy is conserved, so too will angular momentum, localized within the material distribution, be conserved. A verification to this effect will be the subject of future research. Assuming that this is the case, we recall how a conserved angular momentum can be compatible with a reduced spin rate. What comes to mind immediately is the familiar elementary physics demonstration in which a person holding bar-bells near his body is set into rotation on a bearing-suspended platform. The spin is reduced when the person moves the bar-bells away from his body, increasing the moment of inertia and thus conserving angular momentum. In like manner, we conjecture that in either scenario, a) or b), or whatever other less ideal case, the system will expand to compensate for the slow-down to conserve angular momentum. Since a truly rigid body is incompatible with relativity, this does not present a problem in principle. While in the bar-bell example, kinetic energy is also conserved, in the present rotating rod analysis, the calculations indicate that the kinetic energy is diminished while the stress energy increases. The most reasonable prediction would be that the system asymptotically approaches a quasi-stationary configuration. It would be very interesting if an analysis through the slow spin rate phase could be performed to determine the nature of the final state.

In the bar-bell analogy, kinetic energy as well as angular momentum is conserved whereas the rod loses kinetic energy. A closer mechanical analogy for our system which has an extra degree of complexity relative to that of the bar-bell example is as follows: consider two masses attached to a spring and encased in a smooth slot holder with traps to keep the masses from moving outwards. The system is set into spinning motion and the traps are programmed to fall away. Then the masses move outwards and the angular velocity and kinetic energy diminish while the stress energy increases with the increasing tension in the spring. The spring stretches out to its ultimate equilibrium length. Angular momentum and total energy are preserved in the process.

To trace the evolution of the Eddington rod through the later phases would appear to be a very interesting (albeit potentially formidable) challenge for future research.

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Dedication: This work is dedicated to the memory of Nathan Rosen.

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- [17] Many years ago, the late J.L. Synge related to the author how Eddington responded to criticism of some of his mathematical techniques: “I don’t care what they say- I can feel it in my bones!” One can only marvel at the power of Eddington’s skeletal receptors.
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